JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 69, 146–152 (1979)

Metric Characterizations of Upper Semicontinuity

SZYMON DOLECKI AND STEFAN ROLEWICZ

Institute of Mathematics, Polish Academy of Sciences, Sniadeckich 8, Warsaw

Submitted by Ky Fan

We characterize upper semicontinuity of multifunctions in terms of upper Hausdorff semicontinuity, measure of non compactness and active boundary. The results are applicable in optimization, theory of best approximation and in metrizability theory.

1. INTRODUCTION

Throughout the paper a multifunction Γ maps a topological space Y to subsets of a topological space X.

We show, that upper semicontinuity of Γ at some point has several consequences expressed in terms of the measure of non compactness and of the active boundary.

Moreover, we present a necessary and sufficient condition for upper semicontinuity, when X is metric and complete. The condition involves upper Hausdorff semicontinuity and a requirement on the measure of non compactness of some sets.

We indicate how our results apply in several branches of mathematics, e.g., metrizability theorems, theory of best approximation, optimization.

2. PRELIMINARIES

We recall that Γ is called *upper semicontinuous* (u.s.c.) *at* y_0, if for any open set Q with $\Gamma y_0 \subset Q$ there is a neighborhood W of y_0 such that $W \subset \{y: \Gamma y \subset Q\}$ (Kuratowski [4], I, p. 173). An equivalent definition is that the inverse multifunction Γ^{-1} ($\Gamma^{-1}x = \{y: x \in \Gamma y\}$) is closed at y_0, i.e., that for any closed set $F \subset X$

$$y_0 \in \overline{\Gamma^{-1}F} \qquad \text{implies} \qquad y_0 \in \Gamma^{-1}F. \tag{1}$$

Let (X, ρ) be a metric space. Γ is said to be *upper Hausdorff semicontinuous* (u.H.s.c.), *at* y_0, if for any $r > 0$ there is a neighborhood W of y_0 such that $W \subset \{y: \Gamma y \subset B(\Gamma y_0, r)\}$, where $B(\Gamma y_0, r) = \bigcup_{z \in \Gamma y_0} \{x: \rho(x, z) < r\}$.

0022-247X/79/050146–07$02.00/0

Certainly, if Γ is u.s.c. at y_0 and X is metrizable, then Γ is u.H.s.c. at y_0 for each metric ρ of X. It may be easily shown that the converse statement is true, provided that Γy_0 is closed. Indeed, suppose that Γ is not u.s.c. at y_0. This signifies the existence of an open set Q ($Q \supset \Gamma y_0$) such that $\Gamma W \backslash Q$ is not empty for all neighborhoods W of y_0. By the Urysohn lemma there is a continuous function d valued in $[0, 1]$ that vanishes on Γy_0 and is equal 1 outside Q. Pick any metric ρ of X. Then $\rho(x, z) + | d(x) - d(z)|$ is an equivalent metric of X for which $B(\Gamma y_0, 1) \subset Q$. This contradicts the upper Hausdorff semicontinuity for all metrics of X.

3. Measure of Non Compactness

For a subset D of a metric space (X, ρ) we define its *measure of non compactness* $\psi(D)$ as the infimum of those $r > 0$ for which D may be split to a finite number of subsets $\{D_i\}_{i=1,\dots,n}$, $\bigcup_{i=1}^{n} D_i = D$ such that $\sup_{v,x \in D_i} \rho(v, x) \leqslant r$ for each i (Kuratowski [4], I, p. 412).

Let Y possess a countable basis $\{W_n\}_{n=1,2,\dots}$ of neighborhoods of y_0, $W_n \supset W_{n+1}$. For a multifunction Γ denote

$$A_n = \Gamma W_n \backslash \Gamma y_0 . \tag{2}$$

1. Lemma. *Let (X, ρ) be a metric space.*
If Γ is u.s.c. *at y_0, then* $\lim_{n \to 0} \psi(A_n) = 0$.

Proof. Suppose that $\lim \psi(A_n) \neq 0$. As $\{A_n\}$ decreases, there is $b > 0$ such that $\psi(A_n) > b$ for all n.

Let x_1 be an arbitrary element of the (nonempty) set A_1. Suppose that we have selected a collection $\{x_1, x_2, \dots, x_n\}$ such that $x_i \in A_i$, $1 \leqslant i \leqslant n$, and such that $\rho(x_i, x_j) > b/2$ as $i \neq j$. There exists $x_{n+1} \in A_{n+1}$ with the property that $\rho(x_i, x_{n+1}) > b/2$ for $i = 1, 2, \dots, n$.

If it were not so, then for any $x \in A_{n+1}$ there would be $i \leqslant n$ with $\rho(x, x_i) \leqslant b/2$, in other words, it would be that $A_{n+1} = \bigcup_{i=1}^{n} D_i$ where $D_i = \{x \in A_{n+1}: \rho(x, x_i) \leqslant b/2\}$ in contradiction with $\psi(A_{n+1}) > b$.

Observe that the set $\{x_i\}_{i=1,2,\dots}$ constructed above is closed. From (2) if follows that $y_0 \notin \Gamma^{-1}\{x_i\}$ but $y_0 \in \overline{\Gamma^{-1}\{x_i\}}$. Consequently Γ is not u.s.c. at y_0 by (1).

4. Active Boundary

Let $\Gamma: Y \to 2^X$. An element x_0 of X is called *active for y_0* (with respect to Γ), if for each neighborhood Q of x_0 and each neighborhood W of y_0

$$Q \cap (\Gamma W \backslash \Gamma y_0) \neq \varnothing . \tag{3}$$

The set of all active elements for y_0 is termed the *active boundary of* Γy_0 and is denoted by Frac Γy_0 (Dolecki [1], [2]). The following obvious formula

$$\operatorname{Frac} \Gamma y_0 = \bigcap_{W \in \mathbb{B}(y_0)} \overline{(\Gamma W \setminus \Gamma y_0)} \tag{4}$$

($\mathbb{B}(y_0)$ stands for a local basis at y_0) shows that Frac Γy_0 is always closed. On the other hand, Frac Γy_0 is disjoint from the interior of Γy_0.

2. Lemma. *Suppose that X is T_2 (Hausdorff) and admits countable local bases and that Y has a countable basis at y_0.*

If Γ is u.s.c. *at y_0, then* Frac $\Gamma y_0 \subset \Gamma y_0$.

Proof. Suppose that there is $x_0 \in \operatorname{Frac} \Gamma y_0 \setminus \Gamma y_0$. Let $\{W_n\}_{n=1,2,\ldots}$ be a neighborhood basis of y_0 and let $\{Q_k\}_{k=1,2,\ldots}$ be a basis of x_0. Pick out a sequence $\{x_n\}_{n=1,2,\ldots}$ with the property that $x_n \in Q_n \cap \Gamma W_n$ and $x_n \notin \Gamma y_0$. This is possible by (3). Since X is a Hausdorff space x_0 is the only accumulation point of $\{x_n\}_{n=1,2,\ldots}$.

The open set $Q = X \setminus \{x_n\}_{n=0,1,2,\ldots}$ includes Γy_0 but none of ΓW, W a neighborhood of y_0, thus Γ is not u.s.c. at y_0.

We may loosely say that the upper semicontinuity of Γ of y_0 entails the "active closedness" of Γy_0. Note that, if Γy_0 is open, then from upper semicontinuity it follows that Γ is stationary at y_0, i.e., there is a neighbourhood W of y_0 such that $\Gamma W = \Gamma y_0$.

As we shall see, upper semicontinuity implies also the "active compactness" of Γy_0.

3. Lemma (Dolecki [1]). *Let X be metrizable and let Y possess a countable local basis at y_0. If Γ is* u.s.c. *at y_0 then the active boundary* Frac Γy_0 *is compact.*

Proof. Let $\{x_1, x_2, \ldots\}$ be an arbitrary countable subset of Frac Γy_0 and let $\{W_1, W_2, \ldots\}$ denote a local basis at y_0. By (3) we may choose a sequence $\{z_n\}_{n=1,2,\ldots}$ such that $z_n \in \Gamma W_n \cap B(x_n, 1/n)$ and $z_n \notin \Gamma y_0$. As $y_0 \notin \Gamma^{-1} z_n$ and $W_n \cap \Gamma^{-1} z_n \neq \varnothing$ for each n, we conclude that $\Gamma^{-1}\{z_n\}_{n=1,2,\ldots}$ is not closed, hence by upper semicontinuity (1) $\{z_n\}_{n=1,2,\ldots}$ is not closed. Therefore $\{z_n\}$ and $\{x_n\}$ have a (common) cluster point x_0.

Since Frac Γy_0 is closed $x_0 \in \operatorname{Frac} \Gamma y_0$. Since X is metrizable Frac Γy_0 is compact being sequentially compact and closed.

5. Characterization Theorems

4. Theorem. *Let (X, ρ) be metric and complete. Let Y fulfil the first countability axiom at y_0.*

Γ is u.s.c. *at y_0, if and only if*

(i) Γ *is* u.H.s.c *at* y_0

(ii) $\lim_{n\to\infty} \psi(A_n) = 0$

(iii) Frac $\Gamma y_0 \subset \Gamma y_0$.

Proof. In view of Lemmas 1 and 2 we should only prove that (i), (ii), (iii) imply upper semicontinuity. By (ii) there is a sequence $\{b_n\}$ tending to zero and such that for each n there is a finite b_n-net of A_n. Denote such a net by Z_n ($\subset A_n$).

By (i) there is a modulus of semicontinuity $\{a_n\}$ (of Γ at y_0) that tends to zero. (A modulus of semicontinuity is a sequence $\{a_n\}_{n=1,2,\dots}$ which satisfies

$$\Gamma W_n \subset B(\Gamma y_0, a_n), \qquad n = 1, 2, \dots .) \tag{5}$$

Therefore we can find a finite $(b_n + a_n)$-net E_n of A_n such that $E_n \subset \Gamma y_0$ and

$$E_n \subset B(Z_n, a_n) \subset B(A_n, a_n). \tag{6}$$

We shall prove that the following set

$$K = \overline{\bigcup_{n=1}^{\infty} E_n} \tag{7}$$

in compact.

Note that Z_n is a finite $(b_n + a_n)$-net in $B(A_n, a_n)$ and that for $m \geqslant n$

$$E_m \subset B(A_n, a_n). \tag{8}$$

For any $\epsilon > 0$ we may find n such that $a_n + b_n < \epsilon$ and thus in view of (8) there is an ϵ-net of $\overline{\bigcup_{i=n}^{\infty} E_i}$ from which K differs by a finite number of elements. Consequently K is compact for X was assumed to be complete.

We shall show that the set ∂K of cluster points of K is equal to the active boundary Frac Γy_0, which combined with (iii) gives that $K \subset \Gamma y_0$.

Observe that the cluster points of $\bigcup_{n=1}^{\infty} Z_n$ and $\bigcup_{n=1}^{\infty} E_n$ are the same. Suppose that $x_0 \in \partial K$ that is for each n $x_0 \in \overline{\bigcup_{n=i}^{\infty} Z_i} \subset \bar{A}_n$, hence in virtue of (2) and (4) $x_0 \in$ Frac Γy_0. On the other hand, if $x_0 \in$ Frac Γy_0 then for each $\epsilon > 0$ and for any n, $B(x_0, \epsilon/2) \cap A_n \neq \varnothing$, hence $B(x_0, \epsilon) \cap Z_n \neq \varnothing$ whenever $a_n < \epsilon/2$.

Another simple property of K is

$$A_n \subset B(K, a_n + b_n). \tag{9}$$

We are now in position to conclude the proof. Let Q be an arbitrary open set including Γy_0. Since $K \subset \Gamma y_0$ and is compact

$$c = \inf_{x \in K} \operatorname{dist}(x, X \backslash Q) > 0 \tag{10}$$

where $\operatorname{dist}(x, A) = \inf\{r: B(x, r) \cap A \neq \varnothing\}$. We have n such that $a_n + b_n < c$ to ascertain that $A_n \subset B(K, a_n + b_n) \subset Q$. Therefore $\Gamma W_n = A_n \cup \Gamma y_0 \subset Q$ and the proof is complete.

The assumption of completeness in the sufficientcy part cannot be dropped (although may concern only certain subsets of X) even if we add the condition that Frac Γy_0 is compact.

5. Example. $X = \mathbb{R}^2 \backslash \{0\}$, $Y = \overline{\{1/n\}}_{n=1,2,\ldots}$, $\Gamma(1/n) = \{(0, (1/n))\}$, $\Gamma 0 = \{(x_1, x_2) \in X; x_2 = 0\}$. Γ is u.H.s.c. at 0, (ii) is fulfilled, Frac Γy_0 is empty, hence it is a compact subset of Γy_0. The open set $Q = \{(x_1, x_2) \in X; x_1 \neq 0\}$ includes $\Gamma 0$ but none of $\Gamma(1/n)$.

From what we have said we may deduce another characterization of upper semicontinuity.

6. Theorem. *Let X be complete metric and let Γ be a closed-valued* u.H.s.c. *(at y_0) multifunction. The following statements are equivalent:*

- (α) *Γ is* u.s.c. *at y_0.*
- (β) *for each closed $K \subset X$, $K \cap \Gamma$ is* u.H.s.c. *at y_0.*
- (γ) *for each open Q, $\bar{Q} \cap \Gamma$ is* u.H.s.c. *at y_0.*

Proof. (α) $\Rightarrow$ (β) in virtue of a theorem of Kuratowski ([4], I, p. 180) that for each closed set F, $F \cap \Gamma$ is u.s.c. if Γ is u.s.c. and closed-valued.

(β) $\Rightarrow$ (γ) trivially.

(γ) $\Rightarrow$ (α) suppose that Γ is not u.s.c. at y_0. Then by Theorem 4 either $\lim_{n\to\infty} \psi(A_n) \geqslant b > 0$ or Frac $\Gamma y_0 \backslash \Gamma y_0 \neq \varnothing$. In the first case we use the proof of Lemma 1 to obtain a closed sequence $\{x_n\}_{n=1,2,\ldots}$ disjoint from Γy_0 but intersecting each $\Gamma W_n \backslash \Gamma y_0$. Since X is metrizable (hence normal) there is an open set $Q \supset \{x_n\}_{n=1,2,\ldots}$ and $\bar{Q} \cap \Gamma y_0 = \varnothing$. The multifunction $\bar{Q} \cap \Gamma$ is not u.H.s.c. at y_0.

In the second case there is a point x_0 and a neighborhood Q of x_0 such that $\bar{Q} \cap \Gamma y_0 = \varnothing$ and $Q \cap \Gamma W_n \neq \varnothing$ for each n. Thus $\bar{Q} \cap \Gamma$ is not u.H.s.c. at y_0.

6. Some Applications

Let $\Gamma y = f^{-1}(y)$, where f is a continuous function from X to Y which is T_1

7. Corollary (Vainstein lemma [6]). *Let X be metrizable and let Y satisfy the first countability axiom at y_0. If for each closed $F \subset X$ $f(F)$ is closed, then* Fr $f^{-1}(y_0)$ *is compact.*

Proof. Since f is continuous the whole topological boundary is active $\mathrm{Fr}\, \Gamma y_0 \subset \mathrm{Frac}\, \Gamma y_0$ by Lemma 2.3 of [2]. In view of (1), $\Gamma y = f^{-1}(y)$ is u.s.c. (at y_0) thus $\mathrm{Fr}\, f^{-1}(y_0)$ is compact by Lemma 3.

Consider a proper subspace X of a normed space Y. We shall denote by Γy the set of best approximations of y by elements of X:

$$\Gamma y = \{x \in X; \| x - y \| = \inf_{z \in X} \| z - y \|\}. \tag{11}$$

8. Corollary (Singer theorem [5], Th. 4.22, p. 56). *Let Γ be given by* (11). *The following statements are equivalent*

(a) Γ *is* u.s.c. *at* y_0.

(b) Γ *is* u.H.s.c. *at* y_0 *and* Γy_0 *is compact.*

Proof. (a) $\to$ (b).

Since Γy_0 is closed and convex it is enough to prove that the whole boundary $\mathrm{Fr}\, \Gamma y_0$ is active (hence compact by Lemma 3).

If $x_0 \in \mathrm{Fr}\, \Gamma y_0$, then for each $\epsilon > 0$ there is $z \in X$ $\| z \| < \epsilon$ such that $x_0 + z \notin \Gamma y_0$ but $x_0 + z \in \Gamma(y_0 + z)$. In fact

$$\begin{aligned} \|(x_0 + z) - (y_0 + z)\| &= \inf_{x \in X} \| x - y_0 \| = \inf_{x \in X} \|(x - z) - y_0 \| \\ &= \inf_{x \in X} \| x - (y_0 + z)\| . \end{aligned}$$

(b) $\to$ (a)

For any open set $Q \supset \Gamma y_0$ there is $\epsilon > 0$ such that $Q \supset B(\Gamma y_0, \epsilon)$, because Γy_0 is compact.

7. Optimization

Let $f : X \to \overline{\mathbb{R}}$. We say that x_0 is a *local minimum* of f, whenever there is a neighborhood Q of x_0 such that $f(x_0) = \inf_{x \in Q} f(x)$.

The *epigraphic multifunction* E_f associated with f is defined by

$$E_f(x) = \{r \in \mathbb{R}; r \geqslant f(x)\}. \tag{12}$$

The active boundary of $E_f(x)$ may be a singleton $\{f(x)\}$ or empty.

9. Proposition. *An element x_0 of X is a local minimum of f, if and only if*

$$\mathrm{Frac}\, E_f(x_0) = \varnothing . \tag{13}$$

Proof. If x_0 is a local minimum then there is a neighborhood Q of x_0 such that $E_f Q \subset E_f(x_0)$ thus $E_f Q \backslash E_f(x_0)$ is empty and the active boundary is empty. On the other hand, from (13) it follows that there is $\epsilon > 0$ and neighborhood Q of x_0 such that $[f(x), f(x_0)) \cap (f(x_0) - \epsilon, f(x_0) + \epsilon) = \varnothing$ for $x \in Q$. Therefore for $x \in Q [f(x), f(x_0))$ must be empty thus $f(x) \geqslant f(x_0)$.

Let us give some attention to the problem of lower semicontinuity of the primal functional $\overline{f\Gamma}$ associated with a function $f\colon X \to \overline{\mathbb{R}}$ and a multifunction $\Gamma\colon Y \to 2^X$:

$$\overline{f\Gamma}(y) = \inf_{x \in \Gamma y} f(x).$$

It is known (e.g. [3]) that if Γ is u.s.c. at y_0 and f is lower semicontinuous (l.s.c.) on Γy_0 then $\overline{f\Gamma}$ is l.s.c. at y_0.

10. THEOREM. *Assume X to be metrizable and complete and let Y fulfil the first countability axiom.*

Let Γ be u.s.c. *at y_0. There is a compact subset K_0 of Γy_0, such that if f is* l.s.c. *on K_0, then $\overline{f\Gamma}$ is* l.s.c. *at y_0.*

Proof. From upper semicontinuity in view of Theorem 4 it follows that (i) through (iii) hold and thus the set K defined in (7) is a compact subset of Γy_0 with the property that for each open $Q \supset K \cap \operatorname{Fr} \Gamma y_0$ there is a neighbourhood W of y_0 such that $\Gamma W \backslash \Gamma y_0 \subset Q$ (see the proof of Theorem 4). Put $K_0 = K \cap \operatorname{Fr} \Gamma y_0$. Take any $\epsilon > 0$. For each $x_0 \in K_0$ there is a neighbourhood Q_{x_0} such such that $f(x) \geqslant f(x_0) - \epsilon$ for $x \in Q_{x_0}$. Such $K_0 \subset \bigcup_{x_0 \in K_0} Q_{x_0} =^{df} Q$ there is a neighbourhood W of y_0 such that $\Gamma W \backslash \Gamma y_0 \subset Q$.

Let $y \in W$. For each $x \in \Gamma y \backslash \Gamma y_0$ there is $x_0 \in K_0$ such that $f(x) \geqslant f(x_0) - \epsilon$, thus $f(x) \geqslant \inf_{x_0 \in K_0} f(x_0) - \epsilon \geqslant \overline{f\Gamma}(y_0) - \epsilon$.

If $x \in \Gamma y \cap \Gamma y_0$, then $f(x) \geqslant \overline{f\Gamma}(y_0) \geqslant \overline{f\Gamma}(y_0) - \epsilon$. By (14), $\overline{f\Gamma}(y) \geqslant \overline{f\Gamma}(y_0) - \epsilon$ and $\overline{f\Gamma}$ is l.s.c. at y_0.

The statement of Theorem 10 may be improved farther for instance, if $\operatorname{Frac} \Gamma y_0 = \Gamma y_0 \backslash \operatorname{Int} \Gamma y_0$, then we may assume that f is l.s.c. on $\operatorname{Frac} \Gamma y_0$. This situation occurs in the mathematical programming with continuous constraints functions. From the construction of K we see that if $\overline{\Gamma y_0} = \overline{\operatorname{Int} \Gamma y_0}$, then also we may take $K_0 = \operatorname{Frac} \Gamma y_0$.

REFERENCES

1. S. DOLECKI, Constraints stability and moduli of semicontinuity, preprint of 2nd IFAC Symposium, Warwick, June 1977.
2. S. DOLECKI, Semicontinuity in constrained optimization, II. Metric spaces, *Contr. Cybernet.* **7** (1978), 5–16.
3. S. DOLECKI AND S. ROLEWICZ, A characterization of semicontinuity-preserving multifunctions, *J. Math. Appl.* **65** (1978), 26–31.
4. K. KURATOWSKI, "Topology" (English trans.), Vol. I and II, Academic Press, New York, and Polish Scientific Publishers, 1966.
5. I. SINGER, "The Theory of Best approximation and Functional Analysis,"SIAM, Philadelphia, 1974.
6. I. A. VAINSTEIN, On closed mappings of metric spaces, *DAN, S.U.* **57** (1947), 419–421 (in Russian).